Correlations of the local density of states in quasi-one-dimensional wires

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We report a calculation of the correlation function of the local density of states in a disordered quasi-one-dimensional wire in the unitary symmetry class at a small energy difference. Using an expression from the supersymmetric sigma model, we obtain the full dependence of the two-point correlation function on the distance between the points. In the limit of zero energy difference, our calculation reproduces the statistics of a single localized wave function. At logarithmically large distances of the order of the Mott scale, we obtain a reentrant behavior similar to that in strictly one-dimensional chains.

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I. INTRODUCTION

Anderson localization in quasi-one-dimensional (Q1D) disordered systems is known as a broad universality class for various problems of contemporary condensed matter physics and of quantum chaos.[1–6] Quasi-one-dimensional wires are characterized by a large number $N \gg 1$ of transverse channels. Starting from the mean free path scale l, electrons propagate diffusively and eventually get localized[7] with the localization length[8] $\xi \sim Nl \gg l$. Quantum diffusion in the broad range of length scales is the key feature of Q1D systems distinguishing them from strictly one-dimensional (1D) disordered chains, where localization occurs immediately at the scale of the mean free path.[9,10] It is this intermediate diffusive regime that makes studying localization in Q1D systems a very challenging problem.

In order to describe the nonperturbative regime of strong localization, a rather sophisticated mathematical technique is required, both in the 1D and Q1D cases. For the strictly 1D geometry, an appropriate technique has been elaborated by Berezinsky,[11,12] and recently it was translated into a field-theoretic language in Ref. 13. In a Q1D geometry, where localization takes place in the process of quantum diffusion, the diffusive supersymmetric σ model introduced by Efetov[1,14] is a powerful tool to describe the system.

Despite the physical difference between the two models, localization in the 1D and Q1D geometries often looks similar. This analogy is most pronounced in the behavior of a *single* localized wave function: the statistics of the smooth wave function envelopes is precisely the same in the 1D (Refs. 15–17) and Q1D (Ref. 18) geometries, while the local statistics of the rapidly oscillating components involves the plane-wave (1D) or random-matrix (Q1D) moments.[19]

On the other hand, comparison between the quantities involving *different* wave functions in the two models is problematic since almost nothing is known about the eigenfunction correlations in the Q1D geometry in the most physically interesting limit of sufficiently small energy difference ω.

At the same time, quite a lot is known about eigenfunction correlations in the strictly 1D geometry under similar conditions. One of the most important results is the expression for the dissipative low-frequency conductivity which follows the Mott-Berezinsky law:[11,20] $\mathrm{Re}\,\sigma(\omega) \propto \omega^2 \ln^2(1/\omega\tau)$, where τ is the elastic time. Another result concerns the behavior of the correlation function

$$R(\omega;\mathbf{r}_1,\mathbf{r}_2) = \nu^{-2}\langle \rho_\varepsilon(\mathbf{r}_1)\rho_{\varepsilon+\omega}(\mathbf{r}_2)\rangle \tag{1}$$

of the local density of states (LDOS) $\rho_\varepsilon(\mathbf{r}) = \Sigma_n |\psi_n(\mathbf{r})|^2 \delta(\varepsilon - \varepsilon_n)$, where ν is the average density of states (for spinless particles). The dependence of $R(\omega;\mathbf{r}_1,\mathbf{r}_2)$ on $|\mathbf{r}_1-\mathbf{r}_2|$ has been addressed for 1D geometry by Gor'kov *et al.*[21] They have identified that in the limit of small energy separation, $\omega\tau \ll 1$, the spatial behavior of the LDOS correlator is determined by two scales: the localization length, $\xi_{1D} \sim l$, and the Mott scale, $L_M^{1D} \sim \xi_{1D} \ln(1/\omega\tau) \gg \xi_{1D}$. The eigenstates are uncorrelated at $r \gg L_M^{1D}$ and exhibit nearly perfect level repulsion at $\xi_{1D} \ll r \ll L_M^{1D}$, which gets reduced at $r \ll \xi_{1D}$. Such a behavior can be qualitatively understood[22] using the Mott's picture[23] of two nearly degenerate localized states separated by the distance L_M^{1D} which hybridize to form a pair of eigenstates effectively contributing to $R(\omega;\mathbf{r}_1,\mathbf{r}_2)$.

In the Q1D geometry, neither the derivation of the low-frequency conductivity nor the full dependence of the LDOS correlation function have been reported. Though the one-dimensional diffusive supersymmetric σ model has been reformulated by Efetov and Larkin[24] as a quantum-mechanical problem, the resulting set of differential equations is still too complicated and resisting a naive perturbative expansion in the limit $\omega \to 0$.

Recently, a significant progress in solving this effective quantum mechanics has been achieved by two of the authors,[25] who found the exact zero mode of the transfer-matrix Hamiltonian for the unitary symmetry class. As a result, a nonperturbative expression for the short-scale (at distances $|\mathbf{r}_1-\mathbf{r}_2| \ll \xi$) behavior of the LDOS correlator [Eq. (1)] has been derived for arbitrary frequencies ω. For the first time, the difference in the correlations of the localized states in the 1D and Q1D geometries has been demonstrated.

The purpose of the present work is to develop a regular approach to the small-ω expansion of the one-dimensional diffusive supersymmetric σ model. We will assume the unitary symmetry class and employ the exact knowledge of the zero mode obtained in Ref. 25. Developing an advanced perturbation theory which treats both powers and logarithms of ω, we will derive the expression [Eq. (65)] for the LDOS correlation function at arbitrary distances $|\mathbf{r}_1-\mathbf{r}_2|$ between the observation points and analyze its behavior in various asymptotic regions.

The LDOS correlation function [Eq. (1)] can generally be written as[25]

$$R(\omega;\mathbf{r}_1,\mathbf{r}_2) = 1 + A(\omega,t) + k(\mathbf{r}_1,\mathbf{r}_2)B(\omega). \tag{2}$$

Here the factor $k(\mathbf{r}_1,\mathbf{r}_2)=\langle \text{Im}\, G^R(\mathbf{r}_1,\mathbf{r}_2)\rangle^2/(\pi\nu)^2$ accounts for short-scale Friedel oscillations.[19,26] It is equal to 1 at coincident points and exponentially decays as $|\mathbf{r}_1-\mathbf{r}_2|$ exceeds the mean free path. The function $A(\omega,t)$ describes long-range correlations, with

$$t = x/\xi \tag{3}$$

measuring the distance $x=x_1-x_2$ between the observation points in units of the localization length,

$$\xi = 2\pi\nu_1 D, \tag{4}$$

where D is the diffusion coefficient and $\nu_1=\nu S$ is the one-dimensional density of states (S is the wire cross section).

The functions $A(\omega,0)$ and $B(\omega)$ which determine local correlations at $x \ll \xi$ have been calculated in Ref. 25,

$$A(\omega,0) = \frac{4}{3}\text{Re}[\kappa^2(I_1^2 - I_0 I_2)(K_1^2 - K_0 K_2) - I_1^2 K_1^2], \tag{5}$$

$$B(\omega) = \frac{4}{3}\text{Re}(I_1 K_1 + 2 I_2 K_0), \tag{6}$$

where $\kappa=\sqrt{-4i\omega/\Delta_\xi}$ and the argument κ of the modified Bessel functions is suppressed.

At large separations exceeding the mean free path, $x \gg l$, the function $k(\mathbf{r}_1,\mathbf{r}_2)$ exponentially decays and Eq. (2) simplifies to

$$R(\omega,x) = 1 + A(\omega,t). \tag{7}$$

The behavior of $R(\omega,x)$ as a function of $t=x/\xi$ calculated with the help of the general expression (66) is shown in Fig. 1. Different curves on the graph correspond to different values of the ratio ω/Δ_ξ, where the level spacing within the localization length,

$$\Delta_\xi = D/\xi^2, \tag{8}$$

is the natural frequency scale in the localization problem.

One can clearly see that in the deeply localized regime, $\omega \ll \Delta_\xi$, the behavior of $R(\omega,x)$ is similar to the behavior in the strictly 1D geometry:[21,22] on increasing x, the function $R(\omega,x)$ first decays at the localization length and then reaches the uncorrelated value $R=1$ at the Mott scale

$$L_M = 2\xi \ln(\Delta_\xi/\omega). \tag{9}$$

The function $R(\omega,x)$ has a minimum of the order of $(\omega/\Delta_\xi)^{1/4}$ at $x \sim L_M/2$.

The paper is organized as follows. In Sec. II, starting with the supersymmetric σ-model formalism, we give the expression of the LDOS correlation function in terms of a matrix element of the evolution operator with the effective Hamiltonian. The bosonic and fermionic eigensystems of this Hamiltonian are perturbatively constructed in Secs. III and IV, respectively. In Sec. V we give the general expression for the LDOS correlator and evaluate it for arbitrary x and small ω. The opposite limit of small x and arbitrary ω is considered in Sec. VI. Section VII is devoted to the discussion of results and comparison between the 1D and Q1D localization. Finally, a large portion of technical details is relegated to several appendixes.

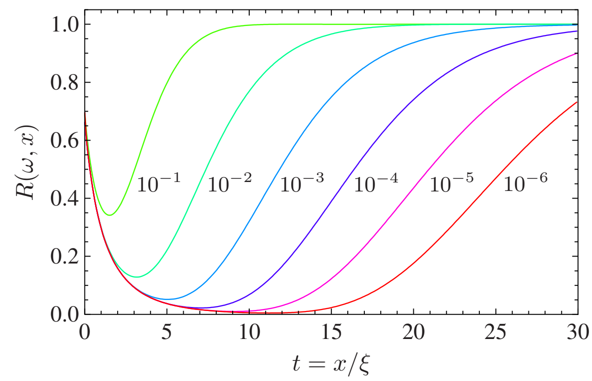

FIG. 1. (Color online) Dependence of the LDOS correlation function $R(\omega,x)$ on $t=x/\xi$ for various values of ω/Δ_ξ shown on the graph.

II. SUPERSYMMETRIC SIGMA MODEL AND GENERAL EXPRESSION FOR THE LDOS CORRELATION FUNCTION

An efficient analytical tool for studying disordered metallic systems—nonlinear supersymmetric σ model—was proposed in the works of Efetov.[1,14] This approach is based on the effective action

$$S[Q] = -\frac{\pi\nu_1}{4}\text{str}\int dx\left[D\left(\frac{dQ}{dx}\right)^2 + 2i\omega\Lambda Q\right] \tag{10}$$

for slow diffusion modes described by the 4×4 supermatrix Q acting in the direct product of the Fermi-Bose (FB) and retarded-advanced (RA) spaces (we assume the unitary symmetry class). The Q matrix is subject to the nonlinear constraint $Q^2=1$. In Eq. (10), $\Lambda=\text{diag}(1,-1)$ the matrix in the RA space.

The one-dimensional σ model [Eq. (10)] can be mapped[24] onto a quantum mechanics of the Q matrix, with the coordinate x along the wire playing the role of time. In that language, the functional integral with the action (10) describes evolution of the wave function $\Psi(Q)$ under the action of an appropriate Hamiltonian $\mathcal{H}$,

$$\frac{\partial}{\partial t}\Psi(Q) = -2\mathcal{H}\Psi(Q), \tag{11}$$

where t is the dimensionless "time" [Eq. (3)] measured in units of the localization length. In the unitary symmetry class, the Q matrix is generally parameterized by four commuting and four Grassmann variables. However in many cases it suffices to consider only the "radially symmetric" ("singlet") wave functions $\Psi(Q)=\Psi(\lambda_B,\lambda_F)$ which depend only on Efetov's coordinates $\lambda_B\in[1,\infty)$ and $\lambda_F\in[-1,1]$, parametrizing the noncompact BB and compact FF sectors.[1,14] On the space of such functions, the Hamiltonian $\mathcal{H}$ acquires the form[1,24]

$$H=-\frac{(\lambda_B-\lambda_F)^2}{2}\left[\frac{\partial}{\partial\lambda_F}\frac{1-\lambda_F^2}{(\lambda_B-\lambda_F)^2}\frac{\partial}{\partial\lambda_F}+\frac{\partial}{\partial\lambda_B}\frac{\lambda_B^2-1}{(\lambda_B-\lambda_F)^2}\frac{\partial}{\partial\lambda_B}\right]+\Omega(\lambda_B-\lambda_F), \tag{12}$$

where we have introduced the dimensionless frequency,

$$\Omega=-\frac{i\omega}{4\Delta_\xi}. \tag{13}$$

At $\omega=0$, Hamiltonian (12) reduces to the Laplace operator on the space of Q matrices. This operator is known[27,28] to have a discrete spectrum in the FF sector and a continuous spectrum in the BB sector. The main technical difficulty in treating the finite-ω case is that any nonzero ω renders the continuous BB spectrum be discrete (assuming Ω positive). This is related to the noncompactness of the BB sector eventually responsible for localization.

The zero mode of Hamiltonian (12) solving $H\Psi_0=0$ has been recently obtained in a closed form valid for all frequencies,[25]

$$\Psi_0(\lambda_F,\lambda_B)=I_0(q)pK_1(p)+qI_1(q)K_0(p), \tag{14}$$

where

$$p=\sqrt{8\Omega(\lambda_B+1)},\quad q=\sqrt{8\Omega(\lambda_F+1)}. \tag{15}$$

The function $A(\omega,t)$ which determines the large-distance behavior of the LDOS correlation function [Eq. (7)] can be conveniently written in terms of the expectation value of a certain evolution operator[1,29] calculated with respect to the zero mode [Eq. (14)],

$$A(\omega,t)=\frac{1}{2}\mathrm{Re}\langle\Psi_0|e^{-2\tilde{H}t}|\Psi_0\rangle, \tag{16}$$

where $\langle\cdot|\cdot\rangle$ is the flat bilinear form on singlet states,

$$\langle\phi|\chi\rangle=\int_{-1}^{1}d\lambda_F\int_0^\infty d\lambda_B\phi(\lambda_B,\lambda_F)\chi(\lambda_B,\lambda_F). \tag{17}$$

The Hamiltonian $\tilde{H}=(\lambda_B-\lambda_F)^{-1}H(\lambda_B-\lambda_F)$ splits into the sum of the bosonic and fermionic parts,

$$\tilde{H}=\tilde{H}_B+\tilde{H}_F, \tag{18}$$

where

$$\tilde{H}_B=-\frac{1}{2}\partial_{\lambda_B}(\lambda_B^2-1)\partial_{\lambda_B}+\Omega\lambda_B, \tag{19}$$

$$\tilde{H}_F=-\frac{1}{2}\partial_{\lambda_F}(1-\lambda_F^2)\partial_{\lambda_F}-\Omega\lambda_F. \tag{20}$$

Analytic calculation of the expectation value in Eq. (16) as a function of arbitrary ω and t is a complicated task. The main technical problem is that the eigenfunctions of the Hamiltonians $\tilde{H}_B$ and $\tilde{H}_F$ (known in the theory as a particular class of the Coulomb spheroidal functions[30]) cannot be found explicitly for an arbitrary value of the parameter Ω.

In the limit $\Omega\gg 1$, only small deviations of λ_F and λ_B from 1 are important, and the correlation function $R(\omega,t)$ can be calculated perturbatively in Δ_ξ/ω. In this regime the LDOS correlations can be found within the standard diagrammatic technique by expanding in the diffusive modes.[31]

In the most interesting case $\Omega\ll 1$, a naive application of the perturbation theory in Ω fails in the "bosonic" sector. Smallness of Ω in the last term in Eq. (19) is compensated by large values of λ_B, and thus the term $\Omega\lambda_B$ in $\tilde{H}_B$ cannot be considered as a small perturbation. An accurate construction of the perturbation theory in the bosonic sector at $\Omega\ll 1$ is a subject of Sec. III.

Though physically relevant Ω is imaginary [see Eq. (13)], we will perform calculations assuming that Ω is a real positive number and make the analytic continuation to imaginary Ω at the final stage of the calculation. With real Ω, both $\tilde{H}_B$ and $\tilde{H}_F$ become Hermitian operators with a discrete spectrum that allows to write the spectral decomposition of the expectation value [Eq. (16)],

$$A(\omega,t)=\frac{1}{2}\mathrm{Re}\sum_{mk}\frac{\langle\Psi_0|\chi_m\phi_k\rangle^2}{\langle\chi_m|\chi_m\rangle\langle\phi_k|\phi_k\rangle}e^{-2(E_m+E_k)t}, \tag{21}$$

where $\chi_m(\lambda_F)$ and $\phi_k(\lambda_B)$ are the eigenfunctions of the Hamiltonians $\tilde{H}_F$ and $\tilde{H}_B$,

$$\tilde{H}_F\phi_m(\lambda_F)=E_m\phi_m(\lambda_F), \tag{22}$$

$$\tilde{H}_B\phi_k(\lambda_B)=E_k\phi_k(\lambda_B). \tag{23}$$

In the next sections we perturbatively construct the eigenfunctions of Hamiltonians (19) and (20) and calculate the matrix elements,

$$\langle\Psi_0|\chi_m\phi_k\rangle=\langle I_0(q)|\chi_m\rangle\langle pK_1(p)|\phi_k\rangle+\langle qI_1(q)|\chi_m\rangle\langle K_0(p)|\phi_k\rangle \tag{24}$$

needed to evaluate Eq. (21).

In the fermionic sector, the perturbation theory is straightforward, while its application in the bosonic sector is much more involved due to its noncompactness. Nevertheless we start with the bosonic sector in Sec. III and use some notations introduced there in dealing with the fermionic sector in Sec. IV.

III. EIGENSYSTEM OF THE "BOSONIC" HAMILTONIAN $\tilde{H}_B$ AT SMALL $\Omega\ll 1$

The main technical problem in studying the noncompact bosonic sector is that it cannot be decomposed into a "regular part" and a perturbation such that (i) the regular part is exactly solvable and (ii) the perturbation is small on the whole interval $1\le\lambda<\infty$ (in this section we denote the bosonic variable λ_B by λ for brevity). Therefore to have a well-controlled perturbation theory one should use different decompositions of Hamiltonian (19) at small and large λ in order to capture the behavior of the wave function near the singular points $\lambda=1$ and $\lambda=\infty$.

At small $\lambda \ll 1/\Omega$, we will decompose the Hamiltonian as $\tilde{H}_B=\tilde{H}_{B1}^0+V_{B1}$ with

$$\tilde{H}_{B1}^0=-\frac{1}{2}\partial_\lambda(\lambda^2-1)\partial_\lambda, \quad V_{B1}=\Omega\lambda, \tag{25}$$

whereas at large $\lambda \gg 1$ we adopt the decomposition $\tilde{H}_B=\tilde{H}_{B2}^0+V_{B2}$ with

$$\tilde{H}_{B2}^0=-\frac{1}{2}\partial_\lambda(\lambda+1)^2\partial_\lambda+\Omega(\lambda+1), \quad V_{B2}=\partial_\lambda(\lambda+1)\partial_\lambda-\Omega. \tag{26}$$

A. Small λ: Legendre scattering states

The Hamiltonian $\tilde{H}_{B1}^0$ has a continuous spectrum,

$$E_k^{(0)}=\frac{k^2}{2}+\frac{1}{8}, \tag{27}$$

(k is a real number) with the eigenfunctions finite at $\lambda=1$ given by

$$L_k^{(0)}(\lambda)=P_{-1/2+ik}(\lambda), \tag{28}$$

where $P_{-1/2+ik}(\lambda)$ is the Legendre function (with such an index it is usually referred to as the conical function). The standard perturbation theory with respect to the perturbation V_{B1} is ill defined since the matrix elements of the perturbation diverge when integrated over the whole semiaxis $\lambda>1$. Nevertheless it is possible to construct the wave function perturbatively in the limit $1<\lambda\ll 1/\Omega$. Using the properties of Legendre functions, one can show that V_{B1} acting on the bare function $L_k^{(0)}(\lambda)$ shifts its "momentum" k by $\pm i$,

$$V_{B1}L_k^{(0)}(\lambda)=\Omega\frac{\left(ik+\frac{1}{2}\right)L_{k-i}^{(0)}(\lambda)+\left(ik-\frac{1}{2}\right)L_{k+i}^{(0)}(\lambda)}{2ik}. \tag{29}$$

Therefore we can search the exact solution as a formal series,

$$L_k(\lambda)=\sum_{n=-\infty}^{\infty}c_n(\Omega,k)L_{k+in}^{(0)}(\lambda). \tag{30}$$

Then acting by the Hamiltonian $\tilde{H}_B$ and using that $L_{k+in}^{(0)}(\lambda)$ is also an eigenfunction of $\tilde{H}_{B1}^0$ we obtain a tridiagonal system of linear equations for the coefficients $c_n(\Omega,k)$,

$$\left[\frac{(k+in)^2}{2}+\frac{1}{8}-E_k\right]c_n+\frac{\Omega\left(ik-n+\frac{1}{2}\right)}{2(ik-n+1)}c_{n-1}+\frac{\Omega\left(ik-n-\frac{1}{2}\right)}{2(ik-n-1)}c_{n+1}=0. \tag{31}$$

Solution (30) can be multiplied by an arbitrary λ-independent constant. To fix this freedom, we assume

$$c_0(\Omega,k)=1 \tag{32}$$

and will normalize the eigenfunction $\phi_k(\lambda)$ later on; see Sec. III D.

The system of Eq. (31) allows to obtain the series expansion of $c_n(\Omega,k)$ in powers of Ω, starting with $\Omega^{|n|}$. In particular,

$$c_1(\Omega,k)=\frac{i\Omega}{2k}-\frac{i(2k^2+7ik-7)\Omega^3}{32(k-i)k(k+i)^2(k+2i)}+\cdots, \tag{33}$$

$$c_2(\Omega,k)=-\frac{(2k+3i)\Omega^2}{16k(k+i)^2}+\cdots, \tag{34}$$

$$c_3(\Omega,k)=-\frac{i(2k+5i)\Omega^3}{96k(k+i)^2(k+2i)}+\cdots. \tag{35}$$

The coefficients with negative indices can be found from

$$c_{-n}(\Omega,k)=c_n(\Omega,-k). \tag{36}$$

The energy E_k also acquires corrections in powers of Ω. Using Eq. (31), it can be expressed in terms of $c_{\pm 1}(\Omega,k)$,

$$E_k=E_k^{(0)}+\frac{\Omega\left(ik+\frac{1}{2}\right)}{2(ik+1)}c_{-1}(\Omega,k)+\frac{\Omega\left(ik-\frac{1}{2}\right)}{2(ik-1)}c_1(\Omega,k). \tag{37}$$

Thus E_k is given by the series in Ω^2,

$$E_k=E_k^{(0)}+\frac{\Omega^2}{4(k^2+1)}+\frac{(5k^2+7)\Omega^4}{44(k^2+1)^3(k^2+4)}+\cdots \tag{38}$$

B. Large λ: Bessel scattering states

The eigenfunctions of the Hamiltonian $\tilde{H}_{B2}^0$ which decay at $\lambda\to\infty$ are expressed in terms of the modified Bessel function of the second kind (MacDonald function),

$$B_k^{(0)}(\lambda)=\frac{K_{2ik}(p)}{p}, \tag{39}$$

where $p=\sqrt{8\Omega(\lambda+1)}$. The corresponding energy is given by Eq. (27). In order to take the perturbation V_{B2} into account we note that its action on $B_k^{(0)}(\lambda)$ is equivalent to shifting the index k by $\pm i$,

$$V_{B2}B_k^{(0)}(\lambda)=\Omega\frac{\left(ik+\frac{1}{2}\right)B_{k-i}^{(0)}(\lambda)+\left(ik-\frac{1}{2}\right)B_{k+i}^{(0)}(\lambda)}{2ik}. \tag{40}$$

[Note the spectacular coincidence with Eq. (29)] Therefore the solution of $\tilde{H}_B B_k(\lambda)=E_k B_k(\lambda)$ can be naturally written as a series

$$B_k(\lambda) = \sum_{n=-\infty}^{\infty} c_n(\Omega,k) B_{k+in}^{(0)}(\lambda), \tag{41}$$

where the coefficients $c_n(\Omega,k)$ coincide exactly with the coefficients of expansion over Legendre functions in Eq. (30), and can be found from the set of linear equations (31). The spectrum E_k as a series in Ω can be found from the same equation; see Eq. (38).

C. Wave functions of the discrete spectrum

The Legendre and Bessel scattering states constructed above provide the solutions to Hamiltonian (19) with the proper behavior at $\lambda \to 1$ and $\lambda \to \infty$, respectively. These solutions exist for arbitrary "wave vectors" k. The wave function $\phi_k(\lambda)$ satisfying both boundary conditions can be found only for a discrete set of k. Such a wave function can be conveniently represented as

$$\phi_k(\lambda) = \begin{cases} L_k(\lambda), & \lambda < \lambda_*, \\ C_k B_k(\lambda), & \lambda > \lambda_*, \end{cases} \tag{42}$$

with some arbitrary λ_* and a suitably chosen factor C_k. Matching the left- and right-scattering states, $L_k(\lambda)$ and $C_k B_k(\lambda)$, at λ_* provides the condition for the allowed values of k.

In the limit $\Omega \ll 1$, determination of the discrete spectrum is simplified by the fact that there exists a broad intermediate region $1 \ll \lambda \ll 1/\Omega$, where both $L_k(\lambda)$ and $B_k(\lambda)$ can be represented by their asymptotic expressions. To find them we write each term in Eqs. (30) and (41) as a superposition of right- and left-moving waves,

$$L_k^{(0)}(\lambda) = L_k^{(0)+} + L_k^{(0)-}, \tag{43}$$

$$B_k^{(0)}(\lambda) = B_k^{(0)+} + B_k^{(0)-}, \tag{44}$$

and represent the right-moving waves $L_k^{(0)+}$ and $B_k^{(0)+}$ by the following hypergeometric series:

$$L_k^{(0)+} = \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(2ik-m)}{m!\,\Gamma^2\left(\frac{1}{2}+ik-m\right)} \left(\frac{\lambda+1}{2}\right)^{-1/2+ik-m}, \tag{45}$$

$$B_k^{(0)+} = \sum_{m=0}^{\infty} \frac{(-1)^m}{4m!} \Gamma(-2ik-m)[2\Omega(\lambda+1)]^{-1/2+ik+m}. \tag{46}$$

The left-moving waves are expressed by the right-moving waves as $L_k^{(0)-} = L_{-k}^{(0)+}$ and $B_k^{(0)-} = B_{-k}^{(0)+}$.

The right-moving waves in the Legendre and Bessel scattering states should be matched by a suitably chosen factor,

$$L_k^+(\lambda) = C_k^+ B_k^+(\lambda). \tag{47}$$

We compute the value of C_k^+ in the following manner. First we substitute the asymptotic expansion Eq. (45) of the Legendre function into Eq. (30). This yields a double series with the terms $\sim(\lambda+1)^{-1/2+ik-m-n}$. Substituting the Bessel asymptotics [Eq. (46)] into the expansion (41) yields another double series with the terms $\sim(\lambda+1)^{-1/2+ik+m-n}$. In order to match these two asymptotic expansions, we pick the terms with $n=-m$ from the Legendre series and with $n=m$ from the Bessel series. This way we obtain two different representations of the coefficient multiplying $(\lambda+1)^{-1/2+ik}$ in the common asymptotic region $1 \ll \lambda \ll 1/\Omega$. They are single series in c_n, that is in powers of Ω. By equating the two expressions we find C_k^+. The other terms in the asymptotic expansion of L_k^+ and B_k^+ will automatically match due to the properties of the coefficients $c_n(\Omega,k)$. Thus we have

$$C_k^+ = \frac{4\Gamma(2ik)(4\Omega)^{1/2-ik}}{\Gamma(-2ik)\Gamma^2(1/2+ik)} e^{i\Phi(k)}, \tag{48}$$

where the phase factor

$$e^{i\Phi(k)} = \frac{\displaystyle\sum_{n=0}^{\infty} c_n(\Omega,-k) \frac{(-1)^n}{n!} \frac{\Gamma(2ik+n)}{\Gamma(2ik)}}{\displaystyle\sum_{n=0}^{\infty} c_n(\Omega,k) \frac{(-1)^n}{n!} \frac{\Gamma(-2ik+n)}{\Gamma(-2ik)}} \tag{49}$$

has a regular series expansion in powers of Ω^2,

$$e^{i\Phi(k)} = 1 + \frac{ik\Omega^2}{4(k^2+1)^2} + \cdots. \tag{50}$$

The waves in the opposite direction, $L_k^-(\lambda)$ and $B_k^-(\lambda)$, are matched by the coefficient,

$$C_k^- = C_{-k}^+. \tag{51}$$

The full function ϕ_k can be matched only if the two coefficients coincide, $C_k^+ = C_k^-$, which is a condition for the discrete spectrum. It can be conveniently formulated as the condition $S(k)=1$ for the scattering matrix $S(k) \equiv C_k^+/C_k^-$,

$$S(k) = S_0(k) e^{2i\Phi(k)}, \tag{52}$$

where

$$S_0(k) = (4\Omega)^{-2ik} \left[\frac{\Gamma(2ik)\Gamma(1/2-ik)}{\Gamma(-2ik)\Gamma(1/2+ik)}\right]^2. \tag{53}$$

Equation $S(k)=1$ determines the values of the wave vector k corresponding to the states of the discrete spectrum. For the low-lying states ($k \lesssim 1$), one finds

$$k \simeq \frac{\pi N_k}{\ln(1/4\Omega)}, \tag{54}$$

with $N_k = 1, 2, \ldots$ labeling the energy level ($N_k=1$ refers to the ground state). According to Eq. (54), the lowest discrete k's are nearly equidistant. This can be naturally seen if we rewrite the Hamiltonian $\tilde{H}_B$ in terms of the angular variable $\theta = \operatorname{arccosh} \lambda$. The motion in the resulting potential will resemble the motion in a box of size $\ln(1/4\Omega)$, leading to the quantization rule (54).

At the points of the discrete spectrum, the two coefficients C_k^+ and C_k^- coincide, and we denote them simply by C_k. A calculation of $C_k^2 = C_k^+ C_k^-$ using Eq. (48) gives

$$C_k = \sigma_k \frac{8\sqrt{\Omega}}{\pi} \cosh \pi k, \quad \sigma_k = (-1)^{N_k}, \tag{55}$$

where N_k is the number of the discrete energy level; see Eq. (54).

Remarkably, expression (55) does not have any corrections in Ω and is exact.

D. Normalization of $\phi_k(\lambda)$

The eigenfunction $\phi_k(\lambda)$ perturbatively constructed above does not have the unit norm $\langle \phi_k | \phi_k \rangle$. Instead, by definition, it is normalized in such a way that its expansion [Eq. (30)] in Legendre functions comes with $c_0(\Omega,k)=1$. Therefore we further need to compute the norm of $\phi_k(\lambda)$. This can be done by writing the function $\phi_k(\lambda)$ in the form (42) and calculating the normalization integrals with $[L_k(\lambda)]^2$ and $[B_k(\lambda)]^2$. This program is performed in the Appendix A leading to a surprisingly simple expression,

$$\langle \phi_k | \phi_k \rangle = -i \frac{\coth \pi k}{2\pi} \frac{\partial \ln S(k)}{\partial k} \sum_{n=-\infty}^{\infty} \frac{[c_n(\Omega,k)]^2}{k+in}, \tag{56}$$

where $S(k)$ is the exact scattering matrix [Eq. (52)].

[Strictly speaking, Eq. (56) has not been proven. It has been verified to the order $O(\Omega^6)$, see Appendix A for discussion. Nevertheless, relying on the anticipated duality between the bosonic and fermionic sectors of a supersymmetric theory, we believe that Eq. (56) is exact.]

E. Overlap with the zero mode

The last ingredient required for the evaluation of general expression (21) is the overlap integral $\langle \Psi_0 | \phi_k \rangle$ with the zero mode Ψ_0. According to Eq. (24), this involves two matrix elements from the bosonic sector: $\langle K_0(p) | \phi_k \rangle$ and $\langle pK_1(p) | \phi_k \rangle$. They are calculated in the Appendix C, and the result has the form

$$\langle K_0(p) | \phi_k \rangle = \sigma_k \pi \sum_{n=-\infty}^{\infty} \frac{c_n(\Omega,k)}{2\sqrt{\Omega}\cosh \pi k}, \tag{57}$$

$$\langle pK_1(p) | \phi_k \rangle = \sigma_k \pi \sum_{n=-\infty}^{\infty} \frac{c_n(\Omega,k)[1+4(k+in)^2]}{4\sqrt{\Omega}\cosh \pi k}. \tag{58}$$

The overall sign factor $\sigma_k = \pm 1$ defined in Eq. (55) will drop from the final expression [Eq. (21)].

IV. EIGENSYSTEM OF THE “FERMIONIC” HAMILTONIAN $\tilde{H}_F$ AT SMALL $\Omega \ll 1$

In the fermionic sector, the standard perturbation theory in small Ω can be easily developed by treating the Ω-dependent term in Hamiltonian (20) as a perturbation. However, instead of doing these routine calculations, one can notice a formal duality between the bosonic and fermionic sectors. By substituting $\lambda_B \mapsto \lambda_F$, one finds $\tilde{H}_B \mapsto -\tilde{H}_F$. Therefore, the fermionic eigenfunctions $\chi_m(\lambda_F)$ can be readily obtained from the Legendre-series expansion [Eq. (30)] in the bosonic sector by imposing the condition of regularity at $\lambda_F=-1$. The latter condition restricts k to $-i(m+1/2)$ with $m=0,1,\ldots$, rendering Legendre functions $P_{-1/2+ik}(\lambda_F)$ be Legendre polynomials $P_m(\lambda_F)$.

Thus the fermionic eigenfunctions can be written with the help of the coefficients $c_n(\Omega,k)$ as

$$\chi_m(\lambda_F) = \sum_{n=-\infty}^{\infty} c_n(\Omega, i(m+1/2)) P_{m+n}(\lambda_F), \tag{59}$$

where we use the symmetry relation (36). Since $P_{-l-1}(x) = P_l(x)$, we can rewrite Eq. (59) as a sum of Legendre polynomials with nonnegative indices,

$$\chi_m(\lambda_F) = \sum_{l=0}^{\infty} \gamma_l(\Omega,m) P_l(\lambda_F), \tag{60}$$

where the coefficients $\gamma_l(\Omega,m)$ are defined as

$$\gamma_l(\Omega,m) = c_{l-m}(\Omega, i(m+1/2)) + c_{-1-l-m}(\Omega, i(m+1/2)). \tag{61}$$

The eigenenergy E_m is given by $-E_k$ [see Eq. (37)] with the substitution $ik \mapsto m+1/2$,

$$E_m = \frac{m(m+1)}{2} - \frac{\Omega(m+1)}{2m+3} c_1(\Omega, i(m+1/2)) - \frac{\Omega m}{2m-1} c_{-1}(\Omega, i(m+1/2)). \tag{62}$$

The wave function (60) is normalized according to

$$\langle \chi_m | \chi_m \rangle = \sum_{l=0}^{\infty} \frac{2}{2l+1} [\gamma_l(\Omega,m)]^2. \tag{63}$$

The overlap integrals of $\chi_m(\lambda_F)$ with $I_0(q)$ and $qI_1(q)$ can be calculated with the help of expansion (60) and the following exact expressions:

$$\langle I_0(q) | P_l(\lambda_F) \rangle = \sqrt{\frac{2l+1}{2}} \frac{I_{2l+1}(4\sqrt{\Omega})}{\sqrt{\Omega}}, \tag{64a}$$

$$\langle qI_1(q) | P_l(\lambda_F) \rangle = \sqrt{\frac{2l+1}{2}} \Omega \frac{\partial}{\partial \Omega} \frac{2I_{2l+1}(4\sqrt{\Omega})}{\sqrt{\Omega}}. \tag{64b}$$

V. BEHAVIOR OF $R(\omega,t)$ AT SMALL ω AND AT ARBITRARY t

A. General expression

Now we are in a position to calculate $A(t)$ with the help of the spectral decomposition (21). Since we are aimed at making the analytic continuation to imaginary Ω, it is desirable to get rid of the sum over discrete spectrum in the bosonic sector existing only for real Ω. This can be done with the help of the identity,

$$\sum_k \cdots = -i \sum_{n=-\infty}^{\infty} \int_0^{\infty} \frac{dk}{2\pi} S^n(k) \frac{\partial \ln S(k)}{\partial k} \cdots, \tag{65}$$

where we utilized the fact that the scattering matrix $S(k)=1$ at the points of discrete spectrum and employed the Poisson resummation formula.

As a result, the factor $\partial \ln S(k)/\partial k$ appearing in Eq. (65) cancels $\partial \ln S(k)/\partial k$ in the normalization [Eq. (56)], and the general expression [Eq. (21)] can be written in the following form:

$$A(\omega,t) = \sum_{n=0}^{\infty} A^{(n)}(\omega,t), \tag{66a}$$

$$A^{(0)}(\omega,t) = 4\pi^2 \,\mathrm{Re} \sum_{m=0}^{\infty} \int_0^{\infty} dk\, k \frac{\tanh \pi k}{\cosh^2 \pi k} M_{mk}(\Omega) e^{-2(E_k+E_m)t}, \tag{66b}$$

$$A^{(n>0)}(\omega,t) = 4\pi^2 \,\mathrm{Re} \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} dk \times k \frac{\tanh \pi k}{\cosh^2 \pi k} S^n(k) M_{mk}(\Omega) e^{-2(E_k+E_m)t}. \tag{66c}$$

In deriving Eq. (66c) we used the symmetry of the scattering matrix, $S^{-n}(k)=S^n(-k)$, which allowed us to extend the integral to the whole real axis.

In Eq. (66), the scattering matrix $S(k)$ is defined in Eq. (52), the energies E_k and E_m are given by Eqs. (37) and (62), and the function $M_{mk}(\Omega,k)$ coming from the matrix elements in Eq. (21) has a regular expansion in powers of Ω which can be calculated with the help of Eqs. (56)–(58), (60), (63), (64a), and (64b):

$$M_{mk}(\Omega) = \frac{\left(\sum_{l=0}^{\infty} \sum_{s=-\infty}^{\infty} \frac{\gamma_l(\Omega,m) c_s(\Omega,k)}{2l+1} \left\{ \left[l + \frac{1}{4} + (k+is)^2 \right] I_{2l}(4\sqrt{\Omega}) + \left[l + \frac{3}{4} - (k+is)^2 \right] I_{2l+2}(4\sqrt{\Omega}) \right\} \right)^2}{4\Omega \left[\sum_{l=0}^{\infty} \frac{1}{2l+1} \gamma_l^2(\Omega,m) \right] \left[\sum_{s=-\infty}^{\infty} \frac{k}{k+is} c_s^2(\Omega,k) \right]}. \tag{67}$$

Equation (65) provide a general expression for the function $A(\omega,t)$ in the limit of small ω.

The term $A^{(0)}(\omega,t)$ given by Eq. (66b) is an even function of Ω responsible for the decay of the LDOS correlations at the localization length scale; see Eq. (70) below.

The terms $A^{(n)}(\omega,t)$ with $n>0$ are given by the integrals [Eq. (66c)] with the strongly oscillating function $S^n(k) \propto S_0^n(k) \propto \Omega^{-2ink}$. Therefore they can be naturally calculated by deforming the integration contour to reach the saddle point,

$$k_*^{(n)} = \frac{in \ln(1/4\Omega)}{t}. \tag{68}$$

In the process of contour deformation one would pick up the contribution of the poles of the integrand in Eq. (66c) located at $k=i/2$, i, $3i/2,\ldots$ The interplay between the contributions of the saddle point and of the poles is responsible for the behavior of the LDOS correlator at the Mott scale L_M given by Eq. (9).

In Fig. 1, we plot the spatial dependence of the LDOS correlation function $R(\omega,x)=1+A(\omega,t)$ calculated numerically with the help of Eq. (65), where it is crucial to take the real part and use the fact that Ω is purely imaginary according to Eq. (13). Different curves on the graph correspond to various values of ω in the deeply localized region, $\omega \ll \Delta_\xi$. The smaller is the ratio ω/Δ_ξ, the faster do the sums in Eq. (65) converge.

The qualitative behavior of the LDOS correlation function in the Q1D geometry coincides with the behavior in the strict 1D geometry;[21,22] on increasing x, the function $R(\omega,x)$ first decays at the localization length and then reaches the uncorrelated value $R=1$ at the Mott scale L_M.

B. Leading order in ω

In this section we evaluate Eq. (65) analytically in the limit of vanishing frequency, $\omega/\Delta_\xi \to 0$. Then the localization length and the Mott scale are well separated and one can obtain simple expressions for $R(\omega,x)$ describing its decay at $x\sim\xi$ and further growth near $x\sim L_M$.

It can be easily seen that the coefficients $\gamma_l(\Omega,m)$ decrease with increasing fermionic "momentum" m: $M_{mk}(\Omega) = O(\Omega^{2m-1})$. Therefore, only the ground state ($m=0$) of the fermionic sector contributes to $A(\omega,t)$ in the limit $\omega\to 0$,

$$M_{0k}(\Omega) = \left(\frac{1}{4\Omega} + 1 \right) (k^2+1/4)^2 + O(\Omega). \tag{69}$$

To the leading order in small ω one can also replace $S(k) \to S_0(k)$, $E_k \to E_k^{(0)}$, $E_m \to E_m^{(0)}$, since corrections to these quantities start with Ω^2.

Consider first the contribution of $A^{(0)}(\omega,t)$. Since $\mathrm{Re}\,\Omega=0$, the "dangerous" term $1/4\Omega$ in Eq. (69) does not contribute and we may safely put $\omega=0$,

$$A^{(0)}(0,t) = 4\pi^2 \frac{\partial^2}{\partial t^2}\int_0^\infty k\, dk \frac{\tanh \pi k}{\cosh^2 \pi k} e^{-(k^2+1/4)t}. \quad (70)$$

The terms $A^{(n)}(\omega,t)$ with $n>0$ can be evaluated by deforming the integration contour to pass through the saddle point [Eq. (68)] and picking up the contributions of poles to be crossed. Since the residue at the pole $ip/2$ is proportional to Ω^{np-1}, only the term with $n=1$ is relevant to the leading order in ω,

$$A^{(1)}(\omega,t) = 4\pi^2 \,\mathrm{Re}\int_{-\infty}^\infty dk\, k \frac{\tanh \pi k}{\cosh^2 \pi k} S_0(k)\left(\frac{1}{4\Omega}+1\right) \times (k^2+1/4)^2 e^{-(k^2+1/4)t} + O(\Omega). \quad (71)$$

The value of the integral in Eq. (71) depends on the relative position between the saddle point $k_*^{(1)}=i(t_M+i\pi)/2t$ and the pole $i/2$. They nearly merge at $t\sim t_M$, where

$$t_M = 2\,\mathrm{Re}\,\ln\frac{1}{4\Omega} = 2\ln\frac{\Delta_\xi}{\omega} \gg 1 \quad (72)$$

corresponds to the Mott scale L_M given by Eq. (9). Thus, within the present technique, the appearance of the Mott scale is related to competition between the pole and the saddle point.

The contribution of the saddle point $k_*^{(1)}$ calculated with exponential accuracy is

$$A^{(1)}(\omega,t)\big|_{\text{s. p. } k_*^{(1)}} \sim \exp\left(-\frac{(t-t_M)^2}{4t}\right). \quad (73)$$

The contribution of the pole $i/2$ to $A^{(1)}(\omega,t)$ remains finite in the limit $\omega\to 0$,

$$A^{(1)}(0,t)\big|_{\text{pole at } i/2} = -1. \quad (74)$$

This pole contribution should be taken into account only for $t<t_M$, since at larger t, Im $k_*^{(1)}<1/2$ and the pole $i/2$ should not be crossed in deforming the integration contour to the saddle point.

Various asymptotic regions for the correlator $R(\omega,x)$ are considered below.

1. Decay of correlations at $t<t_M/2$

At spatial separations x smaller than the Mott scale, $t<t_M$, the LDOS correlation function (in the leading order in ω) is determined by the interplay of $A^{(0)}(0,t)$ given by Eq. (70), and the two contributions to $A^{(1)}(\omega,t)$ [Eqs. (73) and (74)]. The pole contribution [Eq. (74)] cancels 1 in Eq. (7). Therefore, $R(\omega,x)$ is determined by the competition between Eqs. (70) and (73). At $t<t_M/2$, the former dominates and the LDOS correlation function reduces to $A^{(0)}(0,t)$ given by Eq. (70),

$$R(0,x) = 4\pi^2 \frac{\partial^2}{\partial t^2}\int_0^\infty k\, dk \frac{\tanh \pi k}{\cosh^2 \pi k} e^{-(k^2+1/4)t}. \quad (75)$$

This function decays at the localization length scale and describes the limiting curve in Fig. 1 as $\omega\to 0$.

Formula (75) is well known in the theory of localization in one-dimensional systems. It describes the decay of a single wave function, both in the 1D (Refs. 15–17) and Q1D (Refs. 1, 18, and 24) geometries. In the two limiting cases of small and large t one finds

$$R(0,x) = \begin{cases} \dfrac{2}{3} - \dfrac{2t}{3} + \dfrac{8t^2}{15} + \cdots, & t \ll 1, \\ \dfrac{\pi^{7/2}}{16t^{3/2}} e^{-t/4}, & t \gg 1. \end{cases} \quad (76)$$

2. Intermediate asymptotics at $t_M/2<t<t_M$

The LDOS correlator decays according to Eq. (75) only for $t<t_M/2$. For larger values of t, the saddle point contribution [Eq. (73)] should be taken into account on the background of the exponentially small $A^{(0)}(0,t)$. They become comparable at $t\approx t_M/2$, which results in a minimum on the $R(\omega,x)$ curve, of the order of $e^{-t_M/8}\sim(\omega/\Delta_\xi)^{1/4}$. At larger distances ($t_M/2<t<t_M$), the LDOS correlator increases as

$$R(\omega,x) \sim \exp\left[-\frac{(t-t_M)^2}{4t}\right]. \quad (77)$$

3. Behavior at the Mott scale, $t\approx t_M$

In the vicinity of the Mott scale, at $|t-t_M|\sim\sqrt{t_M}$, the saddle point $k_*^{(1)}$ approaches $i/2$, and it becomes impossible to treat the pole at $k=i/2$ independently of the saddle point. To evaluate Eq. (71) in this region we note that its first derivative over t does not have a pole at $k=i/2$ and thus can be easily found with the steepest descent method,

$$\frac{\partial A^{(1)}(\omega,t)}{\partial t} = \frac{1}{2\sqrt{\pi t_M}}\exp\left(-\frac{(t-t_M)^2}{4t_M}\right). \quad (78)$$

Integrating (78) in the vicinity of t_M, we get

$$R(\omega,x) = \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{t-t_M}{2\sqrt{t_M}}\right). \quad (79)$$

Equation (79) describes the crossover from $R=0$ to the uncorrelated value $R=1$ at the Mott scale. This crossover is characterized by a sufficiently narrow width $\sqrt{t_M}\ll t_M$. Formula (79) exactly coincides with the result for the strictly 1D geometry obtained in Ref. 21.

4. Beyond the Mott scale, $t\gg t_M$

Finally, we address the behavior at scales much larger than the Mott length. Here $R\approx 1$, and one is interested in the small correction $R(\omega,x)-1$.

In the region $t_M\ll t\ll t_M^2$, $A(\omega,x)$ reduces to $A^{(1)}(\omega,x)$ which is entirely determined by the saddle point $k_*^{(1)}$, and we get

$$R(\omega,x) = 1 - \frac{\pi^{9/2}t_M^3}{32t^{7/2}}\exp\left(-\frac{(t-t_M)^2}{4t}\right). \quad (80)$$

Equation (80) breaks down at longest distances, $t\gg t_M^2$. Here the contribution of the saddle points $k_*^{(n)}$ with $n>1$ become equally important, indicating that the replacement [Eq. (65)] of the sum over discrete levels by an integral is not adequate. Indeed, at $t \gg t_M^2$ the discrete energy levels in the bosonic sector are completely resolved; see Eq. (54). The main correction to $R(\omega,x)=1$ comes from the ground state with $k \simeq \pi/\ln(1/4\Omega)$,

$$R(\omega,x) = 1 + 8\pi^3 \,\mathrm{Re} \frac{ik \tanh \pi k}{\cosh^2 \pi k} \frac{M_{0k}(\Omega) e^{-(k^2+1/4)t}}{\partial \ln S(k)/\partial k}. \quad (81)$$

The scattering matrix for such a small k is $S(k) \approx S_0(k) \approx e^{2ik\ln(1/4\Omega)}$. Using the matrix element from Eq. (69) and performing analytic continuation, we obtain

$$R(\omega,x) = 1 - \frac{2\pi^6}{t_M^3} \sin\left(\frac{8\pi^3 t_M t}{(t_M^2+\pi^2)^2}\right) \times \exp\left(-\frac{t}{4} + \frac{t_M}{2} - \frac{4\pi^2(t_M^2-\pi^2)t}{(t_M^2+\pi^2)^2}\right). \quad (82)$$

In the region $t_M^2 \ll t \ll t_M^4$, Eq. (82) simplifies to

$$R(\omega,x) = 1 - \frac{2\pi^6}{t_M^3} \sin\left(\frac{8\pi^3 t}{t_M^3}\right) e^{-t/4 + t_M/2 - 4\pi^2 t/t_M^2}. \quad (83)$$

Oscillations of the difference $R(\omega,x)-1$ with x can be easily obtained in the opposite limit of large frequencies, $\omega \gg \Delta_\xi$, where localization effects are weak and the standard perturbation theory can be applied. In this regime, $R(\omega,x)-1$ simultaneously decays and oscillates with the characteristic scale given by the diffusive length $\sqrt{D/\omega}$. At present, we do not know any explanation of the fact that these oscillations persist well into the localized region or of the physical origin of the oscillation period scaling as t_M^3.

C. ω^2 correction to $R(\omega,t)$

Since to the leading order in small ω the result [Eq. (75)] coincides with its 1D analog, it is instructive to study the subleading corrections which would reveal the difference with the strictly 1D situation. Such a difference has already been seen[25] in the limit of small spatial separations, $t \ll 1$, and now we calculate the correction to Eq. (75) at arbitrary t.

At sufficiently small t, the function $A(\omega,t)$ can be formally expanded in powers of ω/Δ_ξ,

$$A(\omega,t) = \sum_{l=0}^{\infty} a_l(t)(\omega/\Delta_\xi)^{2l}, \quad (84)$$

where the coefficients $a_l(t)$ are polynomials of $\ln(\Delta_\xi/\omega)$ of order two. The series (84) is asymptotic: at $t \sim t_M/(2l+1)$ the lth term in the sum acquires a nonanalytic dependence on ω (the upturn [Eq. (79)] at the Mott scale is an example of such a behavior for $l=0$).

To the leading order we have

$$a_0(t) = -1 + A^{(0)}(0,t), \quad (85)$$

where $A^{(0)}(0,t)$ is given by Eq. (70).

The subleading coefficient $a_1(t)$ can be obtained from general expression (65) by adding the contributions from $A^{(0)}(\omega,t)$ and $A^{(n>0)}(\omega,t)$. The latter are calculated by deforming the integration contour to the upper half plane and picking the residues at the relevant poles specified in Table I. The resulting expression has the form

TABLE I. The poles of the integrand in Eq. (66c) contributing to the coefficient $a_1(t)$ in front of the ω^2 term in Eq. (84). The index n labels the power of the S matrix, and the index m labels the fermionic state in the matrix-element block M_{mk}.

Pole	n	m	Exponent
$i/2$	1	0	1
	2	0	1
	3	0	1
	1	1	e^{-2t}
i	1	0	$e^{3t/4}$
$3i/2$	1	0	e^{2t}

$$a_1(t) = \left[\frac{(2\mathcal{L}-3t)^2}{24} + \frac{17\mathcal{L}-24t}{18} + \frac{829}{432} - \frac{\pi^2}{24}\right] e^{2t} - \frac{9\pi^4}{512} e^{3t/4} + \frac{43}{72} + \left[\frac{(2\mathcal{L}-t)^2}{24} + \frac{3\mathcal{L}-t}{6} + \frac{7}{16} - \frac{\pi^2}{24}\right] e^{-2t} - \frac{\pi^2}{4}\int_0^\infty dk\, k \frac{\tanh \pi k}{\cosh^2 \pi k}\left(\frac{(k^2+1/4)^2(40k^4+56k^2+7)}{18(k^2+1)^2} + \frac{2(k^2+1/4)^3}{3(k^2+1)} t\right) e^{-(k^2+1/4)t}, \quad (86)$$

where

$$\mathcal{L} = \ln(\Delta_\xi/\omega) - 2\gamma = \frac{t_M}{2} - 2\gamma, \quad (87)$$

and $\gamma = 0.577\ldots$ is the Euler's constant.

Equation (86) is valid for $t < t_M/3$ and provides the exact distance dependence of the ω^2 correction to the main frequency-independent part $R(0,x)$ calculated in Sec. V B, see Eq. (75). At $t \sim t_M/3$, the saddle point $k_*^{(1)}$ goes below the pole $3i/2$, annihilating its contribution at $t > t_M/3$. At longer distances, the leading frequency-dependent correction is due to the saddle point in $A^{(1)}(\omega,t)$ and is no longer proportional to ω^2.

D. Summary

In Table II we collect the results for various contributions to $R(\omega,t)$ obtained in Secs. V B and V C. Besides that, the decay at the localization length scale and the behavior at the Mott scale are given by Eqs. (75) and (79), respectively.

VI. BEHAVIOR OF $R(\omega,t)$ AT SMALL t AND AT ARBITRARY ω

Here we present an alternative approach to calculating $A(\omega,t)$ which does not require the knowledge of the eigensystem of the Hamiltonian $\tilde{H}$. This approach applies to the case of small t and arbitrary ω and amounts to evaluating the expansion of $A(\omega,t)$ in powers of t.

We expand the evolution operator $e^{-2\tilde{H}t}$ in general expression (16) for $A(\omega,t)$ in series over t,

TABLE II. Summary of the results for $R(\omega,x)$ at small frequencies and arbitrary distances, $t=x/\xi$.

Distance	ω-independent term	ω-dependent term ($t_M=2\ln(\Delta_\xi/\omega)\gg 1$)
$1\ll t<\frac{t_M}{3}$	$\frac{\pi^{7/2}}{16t^{3/2}}\exp\left(-\frac{t}{4}\right)$	$\frac{\omega^2}{24\Delta_\xi^2}(t_M-3t)^2\exp(2t)$
$\frac{t_M}{3}<t<t_M$	$\frac{\pi^{7/2}}{16t^{3/2}}\exp\left(-\frac{t}{4}\right)$	$\sim\exp\left(-\frac{(t-t_M)^2}{4t}\right)$
$t_M\ll t\ll t_M^2$	1	$-\frac{\pi^{9/2}t_M^3}{32t^{7/2}}\exp\left(-\frac{(t-t_M)^2}{4t}\right)$
$t_M^2\ll t$	1	$-\frac{2\pi^6}{t_M^3}\sin\left(\frac{8\pi^3t_Mt}{(t_M^2+\pi^2)^2}\right)\exp\left(-\frac{t}{4}+\frac{t_M}{2}-\frac{4\pi^2(t_M^2-\pi^2)t}{(t_M^2+\pi^2)^2}\right)$

$$A(\omega,t)=\sum_{n=0}^{\infty}A_n(\omega)t^n, \tag{88}$$

where

$$A_n(\omega)=\frac{1}{2}\frac{(-1)^n}{n!}\mathrm{Re}\langle\Psi_0|(2\tilde{H})^n|\Psi_0\rangle. \tag{89}$$

It is convenient to represent the matrix elements in the above expression in terms of the variables p and q defined in Eq. (15). The Hamiltonian $\tilde{H}$ [Eq. (18)] acquires the following form in this representation:

$$\tilde{H}=\frac{1}{8}\left[\frac{1}{p}\partial_p p(\kappa^2-p^2)\partial_p+\frac{1}{q}\partial_q q(q^2-\kappa^2)\partial_q+p^2-q^2\right], \tag{90}$$

where $\kappa^2=16\Omega=-4i\omega/\Delta_\xi$. This operator is defined in the region $0<q<\kappa$, $p>\kappa$.

The ground-state wave function Ψ_0 [see Eq. (14)] contains modified Bessel functions of the arguments p and q with indices 0 and 1 only. The result of the action of $\tilde{H}$ on such a function can be reduced to a combination of the same Bessel functions multiplied by some polynomials in p and q. We arrange the resulting expression in the form,

$$(2\tilde{H})^n|\Psi_0\rangle=\sum_{i,k=0}^{1}q^ip^kc_{ik}^{(n)}(p,q,\kappa)I_i(q)K_k(p). \tag{91}$$

The matrices $c^{(n)}$ are even polynomials of q, p, and κ. Their first entries are

$$c^{(0)}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

$$c^{(1)}=\begin{pmatrix}p^2+q^2-\kappa^2 & 0\\ 0 & 1\end{pmatrix},$$

$$c^{(2)}=\begin{pmatrix}-2(p^2-q^2) & \frac{3}{2}(p^2+q^2)-2\kappa^2\\ \frac{3}{2}(p^2+q^2)-2\kappa^2 & 0\end{pmatrix}. \tag{92}$$

To find $A_n(\omega)$, we have to multiply Eq. (91) by Ψ_0 and integrate with $d\lambda_F=q\,dq/4\Omega$ and $d\lambda_B=p\,dp/4\Omega$. The integrals of Bessel functions arising in this calculation reduce to the products of modified Bessel functions and polynomials, owing to the fact that the matrix $c^{(n)}$ contains only even powers of p and q. Thus we can calculate the functions $A_n(\omega)$ up to any n recursively. The first three terms of the expansion (88) are $A_0(\omega)=A(\omega,0)$ given by Eq. (5), $A_1(\omega)=B(\omega)$ given by Eq. (6), and

$$A_2(\omega)=\frac{4}{3}+\frac{2}{15}\mathrm{Re}[(3\kappa^4-11\kappa^2+16)(I_1^2K_0^2+I_0^2K_1^2)+4\kappa(\kappa^2-4)(I_1K_0-I_0K_1)(I_0K_0+I_1K_1)-3(\kappa^4-3\kappa^2+4)I_1^2K_1^2-\kappa^2(3\kappa^2-8)I_0^2K_0^2], \tag{93}$$

where we omit the argument κ of the Bessel functions for brevity.

In the limit $\omega\ll\Delta_\xi$, we can expand the cumbersome exact expressions for $A_n(\omega)$ in powers of ω. The first two terms of this expansion are [$\mathcal{L}$ is defined in Eq. (87)],

$$A_0(\omega)=-\frac{1}{3}+\frac{\omega^2}{36\Delta_\xi^2}(12\mathcal{L}^2+52\mathcal{L}+43-3\pi^2), \tag{94}$$

$$A_1(\omega)=-\frac{2}{3}+\frac{2\omega^2}{9\Delta_\xi^2}(\mathcal{L}+1), \tag{95}$$

$$A_2(\omega)=\frac{8}{15}+\frac{\omega^2}{18\Delta_\xi^2}(12\mathcal{L}^2+40\mathcal{L}+41-3\pi^2). \tag{96}$$

The same terms naturally appear in the expansion of $R(\omega,t)$ [Eqs. (85) and (86)] in powers of small t.

To make sure that the approach developed in this section is completely equivalent to the one adopted in the rest of the paper, we have calculated the first 15 coefficients $A_n(0)$ and verified that they coincide with the coefficients obtained by expanding Eq. (70) in a series in t.

VII. DISCUSSION

In this work, we have developed a general approach to calculating the correlation function of the LDOS in Q1D disordered wires in the unitary symmetry class using the quantum-mechanical reformulation of the supersymmetric σ model. Our approach is convenient both for numerical evaluation of the correlation function (see Fig. 1) and for analytical treatment of various asymptotic regions.

To the leading order in small ω, we demonstrate that the decay of $R(\omega,x)$ at the localization length and its jump to the uncorrelated value of 1 at the Mott scale are described by exactly the same formulae in Q1D and strictly 1D geometries (with the natural replacement $\xi \to \xi_{1D}$ and $L_M \to L_M^{1D}$). However already the next-to-leading correction at small ω are different:[25] $\omega^2 \ln^2 \omega$ in the Q1D geometry and $\omega^2 \ln \omega$ in the 1D geometry.

We believe that the difference between the subleading terms in the 1D and Q1D problems should be attributed to the short-scale structure of the wave functions. In this case, the ω dependence of the subleading terms should be sensitive to the symmetry of the Q1D problem, with the orthogonal case being different from the unitary case considered in the present work.

In the limit of very large spatial separations, $x \gg L_M$, we identify a number of super-Mott scales. At $x \sim \xi \ln^2(\Delta_\xi/\omega)$, the asymptotic behavior of $R(\omega,x)$ changes its functional dependence, leading to a faster decrease in correlations. Even a larger super-Mott scale, $\xi \ln^3(\Delta_\xi/\omega)$, gives the period of decaying oscillations in $R(\omega,x)-1$, according to Eq. (83). The physical origin of these super-Mott scales still remains to be clarified. Note that the very-large-x asymptotics of the LDOS correlator in the 1D case has not been addressed in Ref. 21, and it is thus impossible to make a comparison between the 1D and Q1D results in this region.

The other interesting feature of the problem is the appearance of the whole hierarchy of sub-Mott scales. Each time the saddle point $k_*^{(n)}$ crosses a pole, a narrow step of the width $\delta x \sim \xi \ln^{1/2}(\Delta_\xi/\omega)$ arises at some rational value of x/L_M. The erf behavior in Eq. (79), originating from the interplay between the saddle $k_*^{(1)}$ and the pole $i/2$ is the strongest of those steps appearing already in the leading order. The amplitudes of other, sub-Mott, steps are proportional to some powers of ω/Δ_ξ and therefore they can be seen only in subleading contributions.

We believe that our technique developed for calculating the LDOS correlation functions can be used to find other low-frequency properties of Q1D systems. The long-standing problem in the field concerns the derivation of the low-frequency dissipative conductivity in Q1D wires. The main technical difficulty compared to the case of the LDOS correlations is that calculation of conductivity involves a much more complicated evolution operator[1,29] compared to Eq. (16).

Finally, it remains a challenging task to generalize our approach from the unitary to other symmetry classes. This is a sophisticated problem since the corresponding quantum-mechanical formulation involves more than two independent variables.[1]

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APPENDIX A: NORMALIZATION OF THE EIGENFUNCTION $\phi_k(\lambda)$

Here we compute the norm of the function $\phi_k(\lambda)$ given by Eq. (42),

$$\langle \phi_k | \phi_k \rangle = \int_1^{\lambda^*} [L_k(\lambda)]^2 d\lambda + C_k^2 \int_{\lambda^*}^{\infty} [B_k(\lambda)]^2 d\lambda, \quad \text{(A1)}$$

where λ^* is an arbitrarily chosen value in the intermediate-asymptotics region $1 \ll \lambda^* \ll 1/\Omega$.

We can expand each of the two integrals in Eq. (A1) as a power series in (λ^*+1), and their sum should be independent of λ^*. So we can keep only zeroth-order terms in (λ^*+1) from the start, to simplify the calculation. We denote the operation of singling out zeroth-order terms (constants and logarithms) by $\hat{Z}_{\lambda^*+1}$.

To compute the normalization in a systematic way, we expand $L_k(\lambda)$ and $B_k(\lambda)$ via Eqs. (30) and (41) and combine the terms as

$$\langle \phi_k | \phi_k \rangle = \sum_{n,m=-\infty}^{\infty} c_n c_m I_{nm}, \quad \text{(A2)}$$

where in

$$I_{nm} = I_{nm}^B + I_{nm}^L, \quad \text{(A3)}$$

we only keep zeroth-order (constants or logarithms) terms in (λ^*+1) in the integrals

$$I_{nm}^B = C_k^2 \hat{Z}_{\lambda^*+1} \int_{\lambda^*}^{\infty} B_{k+in}^{(0)}(\lambda) B_{k+im}^{(0)}(\lambda) d\lambda, \quad \text{(A4)}$$

$$I_{nm}^L = \hat{Z}_{\lambda^*+1} \int_1^{\lambda^*} L_{k+in}^{(0)}(\lambda) L_{k+im}^{(0)}(\lambda) d\lambda. \quad \text{(A5)}$$

There are two types of such terms: diagonal ($n=m$) and off-diagonal ($n \neq m$). They have different structure and are computed by different methods.

1. Diagonal terms

Diagonal terms give rise to contributions scaling as $\log \Omega$. They are computed with the use of the normalization lemma of Appendix B.

For the case of Bessel functions, the integral (A4) at $n=m=0$ is written as

$$\hat{Z}_{\lambda^*+1} \int_{\lambda^*}^{\infty} [B_k^{(0)}]^2 d\lambda = \int_{-1+\epsilon}^{\infty} [B_k^{(0)}]^2 d\lambda - \hat{Z}_{\lambda^*+1} \int_{-1+\epsilon}^{\lambda^*} [B_k^{(0)}]^2 d\lambda. \quad \text{(A6)}$$

We imply the limit $\epsilon \to 0$ so that the regular part (with oscillating terms omitted) of the first term may be calculated using the normalization lemma in the Appendix B,

$$\mathrm{Reg}\int_{-1+\epsilon}^{\infty}[B_k^{(0)}]^2 d\lambda = \frac{\Gamma(2ik)\Gamma(-2ik)}{16\Omega} \times\left[-\ln(2\Omega\epsilon) + \frac{1}{2i}\frac{\partial}{\partial k}\ln\frac{\Gamma(2ik)}{\Gamma(-2ik)}\right]. \tag{A7}$$

The second integral in Eq. (A6) is calculated using the expansion of the Bessel functions [Eq. (46)] at $\Omega(\lambda+1)\ll 1$. We again omit all oscillating terms and obtain

$$\hat{Z}_{\lambda^*+1}\mathrm{Reg}\int_{-1+\epsilon}^{\lambda^*}[B_k^{(0)}]^2 d\lambda = \frac{\Gamma(2ik)\Gamma(-2ik)}{16\Omega}\ln\frac{\lambda^*+1}{\epsilon}. \tag{A8}$$

Adding the two contributions [Eqs. (A7) and (A8)] cancels the dependence on ε as it should be and gives

$$I_{00}^B = C_k^2\frac{\Gamma(2ik)\Gamma(-2ik)}{16\Omega} \times\left[-\ln[2\Omega(\lambda^*+1)] + \frac{1}{2i}\frac{\partial}{\partial k}\ln\frac{\Gamma(2ik)}{\Gamma(-2ik)}\right]. \tag{A9}$$

We repeat the same procedure for the Legendre part,

$$\hat{Z}_{\lambda^*+1}\int_1^{\lambda^*}[L_k^{(0)}]^2 d\lambda = \int_1^{\Lambda}[L_k^{(0)}]^2 d\lambda - \hat{Z}_{\lambda^*+1}\int_{\lambda^*}^{\Lambda}[L_k^{(0)}]^2 d\lambda, \tag{A10}$$

with $\Lambda\to\infty$. Applying the normalization lemma from the Appendix B, we find the regular (without oscillations) part of the first term,

$$\mathrm{Reg}\int_1^{\Lambda}[L_k^{(0)}]^2 d\lambda = \frac{\Gamma(2ik)\Gamma(-2ik)}{\Gamma^2(ik+1/2)\Gamma^2(-ik+1/2)} \times 4\left[\ln\frac{\Lambda+1}{2} + \frac{1}{2i}\frac{\partial}{\partial k}\ln\frac{\Gamma(2ik)\Gamma^2(-ik+1/2)}{\Gamma(-2ik)\Gamma^2(ik+1/2)}\right]. \tag{A11}$$

In calculating the regularized second term of Eq. (A10) we use the expansion [Eq. (45)] and obtain

$$\hat{Z}_{\lambda^*+1}\mathrm{Reg}\int_{\lambda^*}^{\Lambda}[L_k^{(0)}]^2 d\lambda = \frac{\Gamma(2ik)\Gamma(-2ik)}{\Gamma^2(ik+1/2)\Gamma^2(-ik+1/2)} \times 4\ln\frac{\Lambda+1}{\lambda^*+1}. \tag{A12}$$

Adding the two terms, we find

$$I_{00}^L = \frac{\Gamma(2ik)\Gamma(-2ik)}{\Gamma^2(ik+1/2)\Gamma^2(-ik+1/2)} \times 4\left[\ln\frac{\lambda^*+1}{2} + \frac{1}{2i}\frac{\partial}{\partial k}\ln\frac{\Gamma(2ik)\Gamma^2(-ik+1/2)}{\Gamma(-2ik)\Gamma^2(ik+1/2)}\right]. \tag{A13}$$

Other diagonal elements I_{nn}^B and I_{nn}^L may be obtained by analytically continuing I_{00}^B and I_{00}^L to $k\to k+in$. Note that such an analytic continuation commutes with the $\hat{Z}_{\lambda^*+1}$ operation.

Finally, we add the Bessel and Legendre contributions together and, using the identity

$$\frac{\Gamma(2ik)\Gamma(-2ik)}{\Gamma^2(ik+1/2)\Gamma^2(-ik+1/2)} = \frac{\coth(\pi k)}{4\pi k}, \tag{A14}$$

Obtain

$$I_{00}(k) = \frac{\coth(\pi k)}{2\pi ik}\frac{\partial}{\partial k}\ln S_0(k), \tag{A15}$$

where $S_0(k)$ is the scattering matrix [Eq. (53)] without corrections in Ω.

The other diagonal terms are given by the analytic continuation,

$$I_{nn}(k) = I_{00}(k+in) = \frac{\coth(\pi k)}{2\pi i(k+in)}\frac{\partial}{\partial k}\ln S_0(k+in). \tag{A16}$$

2. Off-diagonal terms

Off-diagonal terms form a series in Ω with positive powers only. They involve overlaps of two functions which solve the same Hamiltonian at different (complex) energies. Such integrals may be conveniently computed by the Wronskian method, which results in

$$I_{nm}^B = C_k^2\hat{Z}_{\lambda^*+1}\frac{(\lambda^*+1)^2}{(k+in)^2-(k+im)^2} \times[\partial_{\lambda^*}B_{k+in}^{(0)}(\lambda^*)B_{k+im}^{(0)}(\lambda^*) - B_{k+in}^{(0)}(\lambda^*)\partial_{\lambda^*}B_{k+im}^{(0)}(\lambda^*)], \tag{A17}$$

and

$$I_{nm}^L = -\hat{Z}_{\lambda^*+1}\frac{(\lambda^*+1)(\lambda^*-1)}{(k+in)^2-(k+im)^2} \times[\partial_{\lambda^*}L_{k+in}^{(0)}(\lambda^*)L_{k+im}^{(0)}(\lambda^*) - L_{k+in}^{(0)}(\lambda^*)\partial_{\lambda^*}L_{k+im}^{(0)}(\lambda^*)]. \tag{A18}$$

These coefficients have the obvious symmetries

$$I_{mn} = I_{nm} = I_{-n,-m}(k\to -k). \tag{A19}$$

By using the expansions [Eqs. (45) and (46)] of the Legendre and MacDonald functions, after some tedious combinatorics, one obtains the following form of the off-diagonal coefficients [we have used the relation (A14) to simplify products of Gamma functions]:

$$I_{nm}^{B} = (-1)^{n+m} I_{nm}^{L} = \frac{\coth(\pi k)}{\pi |n-m| \left(k + i \frac{n+m}{2} \right)}, \quad \text{(A20)}$$

which leads to

$$I_{nm} = \begin{cases} 0, & m+n \equiv 1 (\mathrm{mod}\ 2), \\ \dfrac{2\coth(\pi k)}{\pi |n-m| \left(k + i \frac{n+m}{2} \right)}, & m+n \equiv 0 (\mathrm{mod}\ 2). \end{cases} \quad \text{(A21)}$$

Thus the Legendre and Bessel contributions cancel each other at odd powers of Ω and double each other at even powers of Ω.

3. Normalization

The diagonal terms [Eq. (A16)] in the sum (A2) contain logarithmic derivatives of $S_0(k+in)$ taken at different n. We may perform two transformation on those terms. First, with the help of gamma-function properties, we reduce the argument of $S_0(k+in)$ to k acquiring new terms which form a series in Ω. Second, we replace $\ln S_0(k)$ by $\ln S(k)$ according to Eq. (52), again at the cost of adding a series in Ω. Surprisingly, these two contributions exactly cancel the off-diagonal part of the sum (A2). We have observed this cancellation up to the order $O(\Omega^6)$ and conjecture that it is exact at all orders.

The resulting expression (after canceling the off-diagonal terms) is proportional to $\partial \ln S(k)/\partial k$ with the full S matrix [Eq. (52)],

$$\langle \phi_k | \phi_k \rangle = -i \frac{\coth \pi k}{2\pi} \frac{\partial \ln S(k)}{\partial k} \sum_{n=-\infty}^{\infty} \frac{[c_n(\Omega,k)]^2}{k+in}. \quad \text{(A22)}$$

APPENDIX B: NORMALIZATION LEMMA

In one dimension, consider a particle moving freely in one asymptotic direction and fully reflecting of a potential in the other direction. Then the normalization of the wave function in the region where the particle experiences reflection may be related to the scattering phase shift.

Specifically, consider the one-dimensional Hamiltonian,

$$H = \frac{p^2}{2m} + U(x) \quad \text{(B1)}$$

with the condition on the potential $U(x)=0$ for $x<0$ and $U(x) \to \infty$ at $x \to \infty$. Then to any positive momentum k there corresponds a state of the continuous spectrum $\Psi_k(x)$ with the asymptotic behavior at $x<0$,

$$\Psi_k(x<0) = e^{ik(x-x_0)} - e^{-ik(x-x_0)+i\varphi_{x_0}(k)}, \quad \text{(B2)}$$

where x_0 is an arbitrarily chosen reference point, and $\varphi_{x_0}(k)$ is the scattering phase relative to this point. Obviously, $\varphi_{x_0}(k)$ is a linear function of x_0 with the slope $2k$. Consider now the normalization of the part of the wave function from x_0 to $+\infty$,

$$I_{x_0,k} = \int_{x_0}^{\infty} |\Psi_k(x)|^2 dx. \quad \text{(B3)}$$

Obviously, for $x_0<0$ this normalization is a linear function of x_0 plus a sinusoidal function. We define the *regularized* normalization integral,

$$\mathrm{Reg}\ I_{x_0,k} = \mathrm{Reg} \int_{x_0}^{\infty} |\Psi_k(x)|^2 dx \quad \text{(B4)}$$

as $I_{x_0,k}$ with the sinusoidal part subtracted.

Then our normalization lemma states that

$$\mathrm{Reg}\ I_{x_0,k} = \frac{\partial \varphi_{x_0}(k)}{\partial k}. \quad \text{(B5)}$$

To prove the lemma, consider the same problem with an infinite wall placed at a position $x_0<0$. Then, for any two points $x,x'>x_0$ we have two decompositions of the delta function: the one by the full basis of the continuous spectrum,

$$\int \frac{dk}{2\pi} \Psi_k^*(x) \Psi_k(x') = \delta(x-x'), \quad \text{(B6)}$$

and the other one by the basis of the discrete spectrum of the states constrained by the infinite wall at x_0,

$$\sum_k (I_{x_0,k})^{-1} \Psi_k^*(x) \Psi_k(x') = \delta(x-x'), \quad x,x' > x_0. \quad \text{(B7)}$$

First, note that in the case of an infinite wall, the normalization of the states of the discrete spectrum coincides with the regularized one: $I_{x_0,k} = \mathrm{Reg}\ I_{x_0,k}$. Second, we may replace

$$\sum_k \to \int \frac{dk}{2\pi} \sum_{n=-\infty}^{+\infty} e^{in\varphi_{x_0}(k)} \frac{\partial \varphi_{x_0}(k)}{\partial k} \quad \text{(B8)}$$

and after subtracting Eq. (B6) from Eq. (B7) we obtain

$$\int \frac{dk}{2\pi} \left[\frac{\partial \varphi_{x_0}(k)}{\partial k} (\mathrm{Reg}\ I_{x_0,k})^{-1} - 1 \right] \Psi_k^*(x) \Psi_k(x') + \sum_{n \neq 0} \int \frac{dk}{2\pi} e^{in\varphi_{x_0}(k)} \frac{\partial \varphi_{x_0}(k)}{\partial k} (\mathrm{Reg}\ I_{x_0,k})^{-1} \Psi_k^*(x) \Psi_k(x') = 0 \quad \text{for any } x,x' > x_0. \quad \text{(B9)}$$

Now we take the limit $x_0 \to -\infty$. In this limit, the first term goes to zero as $1/|x_0|$, since both $\partial \varphi_{x_0}(k)/\partial k$ and $\mathrm{Reg}\ I_{x_0,k}$ are linear functions of x_0. The second term goes to zero faster than $1/|x_0|$, because of the rapidly oscillating exponent. Therefore, the first term must identically vanish, which implies the result (B5).

APPENDIX C: EVALUATION OF $\langle K_0|\phi_k\rangle$ AND $\langle pK_1|\phi_k\rangle$

Let us start with the matrix element $\langle K_0|\phi_k\rangle$. Using the perturbative expansions [Eqs. (30) and (41)] of the wave function ϕ_k, we represent the matrix element in the form

$$\langle K_0|\phi_k\rangle = \sum_{n=0}^{\infty} c_n(\Omega,k)\left[\int_1^{\lambda^*} d\lambda L_{k+in}^{(0)}(\lambda)K_0(p) + C_k\int_{\lambda^*}^{\infty} d\lambda B_{k+in}^{(0)}(\lambda)K_0(p)\right], \tag{C1}$$

with $1 \ll \lambda^* \ll 1/\Omega$. We then use the hypergeometric series [Eqs. (45) and (46)] for Legendre and Bessel functions and integrate every term of these expansions with $K_0(p)$ with the help of the following indefinite integral:

$$\int dp\, p^a K_0(p) = \frac{p^{1+a}K_0(p)}{1+a}\,{}_1F_2\left(1;\frac{1+a}{2},\frac{3+a}{2};\frac{p^2}{4}\right) + \frac{p^{2+a}K_1(p)}{(1+a)^2}\,{}_1F_2\left(1;\frac{3+a}{2},\frac{3+a}{2};\frac{p^2}{4}\right). \tag{C2}$$

The parameter k satisfies the quantization condition $S(k)=1$, which ensures that the two expansions of the wave function in Eq. (C1) match at the point λ^*. This allows us to omit the dependence of the two integrals in Eq. (C1) on λ^* by applying $\hat{Z}_{\lambda^*+1}$ operation defined in the Appendix A.

Let us start with the Legendre part of Eq. (C1). We integrate every term of the series (45) for $L_k^{(0)}$ with $K_0(p)$ using Eq. (C2) and expand the emerging hypergeometric functions. The contribution of the upper limit λ^* does not contain any constant or logarithmic terms in λ^*+1 and therefore is totally annihilated by the $\hat{Z}_{\lambda^*+1}$ operation. At the lower limit $\lambda=1$, we have

$$\hat{Z}_{\lambda^*+1}\int_1^{\lambda^*} d\lambda L_k^{(0)}(\lambda)K_0(p) = \sum_{m,l=0}^{\infty} \frac{\Gamma(2ik-m)(-1)^{m+1}(4\Omega)^l}{\Gamma^2\left(\frac{1}{2}+ik-m+l\right)m!}\left[\frac{2K_0(4\sqrt{\Omega})}{\frac{1}{2}-ik+m-l} + \frac{4\sqrt{\Omega}K_1(4\sqrt{\Omega})}{\left(\frac{1}{2}-ik+m-l\right)^2}\right] + \{k\mapsto -k\}. \tag{C3}$$

We first sum over the index m and obtain

$$\hat{Z}_{\lambda^*+1}\int_1^{\lambda^*} d\lambda L_k^{(0)}(\lambda)K_0(p) = \sum_{l=0}^{\infty} \frac{i\pi(2l)!\,(4\Omega)^l}{\sinh(2\pi k)\,\Gamma^2\left(\frac{1}{2}+ik+l\right)\Gamma^2\left(\frac{1}{2}-ik+l\right)}\left\{\frac{2K_0(4\sqrt{\Omega})}{\left(\frac{1}{2}+n\right)^2+k^2} + \frac{(2l+1)4\sqrt{\Omega}K_1(4\sqrt{\Omega})}{\left[\left(\frac{1}{2}+n\right)^2+k^2\right]^2}\right\} + \{k\mapsto -k\}. \tag{C4}$$

Every term in the above sum over l is an odd function of k and cancels against its $(k\mapsto -k)$ counterpart. This immediately yields zero,

$$\hat{Z}_{\lambda^*+1}\int_1^{\lambda^*} d\lambda L_k^{(0)}(\lambda)K_0(p) = 0. \tag{C5}$$

This identity also holds after the shift of the parameter $k\mapsto k+in$ because the integral remains convergent in the limit $\lambda\to 1$ and the operation $\hat{Z}_{\lambda^*+1}$ commutes with the imaginary momentum shift.

Now we turn to the second integral in Eq. (C1) and rewrite it in the form

$$\hat{Z}_{\lambda^*+1}\int_{\lambda^*}^{\infty} d\lambda B_k^{(0)}(\lambda)K_0(p) = \int_{-1}^{\infty} d\lambda B_k^{(0)}(\lambda)K_0(p) - \hat{Z}_{\lambda^*+1}\int_{-1}^{\lambda^*} d\lambda B_k^{(0)}(\lambda)K_0(p). \tag{C6}$$

The first integral (running from -1 to ∞) is known to be

$$\int_{-1}^{\infty} d\lambda B_k^{(0)}(\lambda)K_0(p) = \frac{\pi^2}{16\Omega\cosh^2(\pi k)}. \tag{C7}$$

In the second integral we will omit the dependence on λ^* according to the $\hat{Z}_{\lambda^*+1}$ prescription. Substituting the expansion of the Bessel function [Eq. (46)] and using Eq. (C2), we see that every term of the hypergeometric series yields zero at the lower limit $p=0$. Therefore the Bessel part of the matrix element is

$$\hat{Z}_{\lambda^*+1}\int_{\lambda^*}^{\infty} d\lambda B_k^{(0)}(\lambda)K_0(p) = \frac{\pi^2}{16\Omega\cosh^2(\pi k)}. \tag{C8}$$

This result holds for the shifted index $k\mapsto k+in$ as well, despite the fact that both integrals in the right-hand side of Eq. (C6) converge only when k is real.

Collecting all the terms we obtain the matrix element

$$\langle K_0|\phi_k\rangle = \frac{\pi^2 C_k}{16\Omega\cosh^2(\pi k)}\sum_{n=0}^{\infty} c_n(\Omega,k). \tag{C9}$$

Calculation of $\langle pK_1|\phi_k\rangle$ is very similar. We separate the Legendre and Bessel parts of the matrix element as in Eq. (C1). With the help of the result [Eq. (C5)] we again find that the Legendre integral yields zero

$$\hat{Z}_{\lambda^*+1}\int_1^{\lambda^*} d\lambda\, L_k^{(0)}(\lambda)pK_1(p) = -2\Omega\frac{\partial}{\partial\Omega}\hat{Z}_{\lambda^*+1}\int_1^{\lambda^*} d\lambda\, L_k^{(0)}(\lambda)K_0(p) = 0. \quad \text{(C10)}$$

Next, we extend the Bessel integral to the whole axis, as we did in Eq. (C6), and find

$$\hat{Z}_{\lambda^*+1}\int_{\lambda^*}^{\infty} d\lambda\, B_k^{(0)}(\lambda)pK_1(p) = \int_{-1}^{\infty} d\lambda\, B_k^{(0)}(\lambda)pK_1(p) = \frac{\pi^2(1+4k^2)}{32\Omega\cosh^2(\pi k)}. \quad \text{(C11)}$$

After the analytic continuation $k\mapsto k+in$ and collecting all the terms, we get

$$\langle pK_1|\phi_k\rangle = \frac{\pi^2 C_k}{32\Omega\cosh^2(\pi k)}\sum_{n=0}^{\infty} c_n(\Omega,k)[1+4(k+in)^2]. \quad \text{(C12)}$$

Using the constant C_k from Eq. (55) we finally obtain the matrix elements in the form Eqs. (57) and (58).

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